Journey Through Genius: The Great Theorems of Mathematics Study Guide

Journey Through Genius: The Great Theorems of Mathematics by William Dunham (mathematician)

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Contents

Plot Summary

Journey Through Genius: The Great Theorems of Mathematics is a survey of twelve great theorems selected by author William Dunham for the importance to the field of mathematics as well as for how they represent the prevailing ideas and ideals of the times in which they appear.

Dunham takes a chronological path through the subject, beginning with the ancient Greeks and the thinkers centered in Alexandria, the site of the greatest collection of learning in the ancient world. Beginning with Hippocrates of Chios, Dunham introduces Euclid, whose geometry text is still taught today, Archimedes, the absent-minded inventor and thinker who accurately estimates the value of π, and Heron, who furthers the analysis of the triangle. These three mathematicians have an impact so great nobody approaches their advances for many centuries.

Dunham picks up the thread in the 16th century with the eccentric and superstitious Gerolamo Cardano who is jailed for heresy at one point but manages to solve equations once thought unsolvable. He pays homage to the great Isaac Newton and his elegant estimation of π. He describes the contentious Bernoulli brothers, Johann and Jakob, who despite their bickering manage to transform the mathematics of their day. Dunham is especially reverent toward Leonhard Euler and Georg Cantor, two incredibly prolific mathematicians who push the boundaries of theoretical math despite physical and mental challenges.

In each chapter, Dunham presents the proof of a "great theorem" by the subject of the chapter surrounded by introductory biographical and historical background to place the theorem in context, as well as an epilogue describing how the great theorem is received and the importance it holds for subsequent thinkers. Dunham's demonstrations are thorough and require careful attention, but do not rely on any advanced knowledge of mathematics to comprehend. Rather than drill the math, Dunham wishes to present the theorems in a setting that enhances their historical importance and it is not necessary to completely comprehend each theorem to grasp their influence.

Dunham also charts the metaphysical aspects of mathematics through the centuries, from the Pythagorean idea that the natural world can be measured in whole units of fixed proportions to Georg Cantor's religious notions about God's role in his exploration of the infinite. He describes the reluctance of mathematicians to divulge their discoveries, sometimes out of fear of having their precious assets taken from them, but also sometimes because of a fear that some more theoretical ideas might not be accepted among their colleagues. Dunham ably describes how theoretical math is often at the forefront, and how ideas that seem too bizarre to publish in one era come to have practical significance in another once science and technology have caught up.

Hippocrates' Quadrature of the Lune

Hippocrates' Quadrature of the Lune Summary and Analysis

Dunham begins with a brief overview of early mathematics. Since the dawn of agriculture humans have had a grasp of measurement and basic geometric concepts like area. The early Egyptians know some of the properties of specific shapes such as right triangles with sides of 3, 4 and 5 units but do not develop the higher theories that examine these relationships.

The first of the ancients to ask "why" is Thales, who lives in Asia Minor around 600 BC. Almost all that is known about Thales comes from later mathematicians who refer to his theories. After Thales, Pythagoras is the next major figure in Greek mathematics. Pythagoras is attributed with proving what is now called the Pythagoraean Theorem, which holds that for any right triangle the square of the diagonal side is equal to the sum of the squares of the two legs ($a^2 = b^2 + c^2$). With Pythagoras, the concept of providing a logical proof for observed geometric properties fully matures.

Hippocrates is a teacher from the island of Chios. He is perhaps the first to take the idea of constructing a geometry from basic elements by proving subsequently more complex theorems based on earlier proofs, as Euclid will do later on. Nothing remains of these proofs, however. they are only known through reference from later mathematicians. Hippocrates is also widely known for his "quadrature of the lune," which is the great theorem Dunham addresses in this chapter.

Quadrature is the determination of the area of an enclosed space by constructing a square of an equivalent area that can be easily measured. This has a practical use in figuring the area of oddly shaped areas such as parcels of land. Before Hippocrates, the Greeks have already learned to "square" rectangles and triangles allowing them to determine the area of nearly any odd shape with straight sides by reducing it to smaller triangles. This can be done using only a compass to draw arcs and circles and an unruled straightedge to draw lines.

Hippocrates' great advance is devising a way to "square" a figure with curved sides, specifically a "lune," or crescent. His theory and proof are fairly simple and are based on properties of triangles and semicircles that are already known. Encouraged by his success, the Greeks begin to wonder if other curved shapes such as a circle might be squared. This question occupies mathematicians for centuries until in 1882 a mathematician named Ferdinand Lindeman proves that is is impossible to square a circle using only a straightedge and compass. Lindeman proves this by showing that some numbers are not "constructable" with only straightedge and compass. One of these numbers is π. Since the area of a circle is related to π, and because this number is impossible to construct geometrically, Lindeman shows that the quadrature of a circle is impossible.

Euclid's Proof of the Pythagorean Theorem

Euclid's Proof of the Pythagorean Theorem Summary and Analysis

Dunham next turns to Euclid in the first of two chapters devoted to the great mathematician. Euclid major contribution to mathematics is his book on geometry and number theory called the Elements. Euclid's Elements, Dunham writes, is a revolutionary book not so much for what it says, for much if what it contains is already known from earlier authors, but in how it is presented. Euclid begins with some very basic definitions and gradually builds on them to higher conclusions in a methodical way, creating an "axiomatic framework" for his proofs that allows for solid proofs based on consistent definitions.

In the first book of his Elements, Euclid offers 23 definitions, beginning with the definition of a point, a line, and a straight line. He also defines right angles, circles, and in what is to become an important definition to later mathematicians, parallel lines. These Euclid defines as straight lines that are in the same plane that will never meet if they are extended forever.

After his definitions, Euclid moves on to five postulates which must be accepted as given, such as that it is possible to connect any two points with a line and to draw a circle. These postulates allow Euclid to construct the diagrams necessary for his proofs. Next he presents five "common notions" such as "Things which are equal to the same thing are also equal to one another," (p. 36) which allow him to move logically from step to step in his proofs. The meat of Euclid's Elements is in the 465 propositions which he makes and then proves. It is the 47th and 48th propositions that address the Pythagorean theorem, and which Dunham focuses on as the great theorems of the chapter.

The Pythagorean theorem is not new when Euclid takes it up. It is already known that the square of the diagonal hypotenuse of a right triangle is equal to the sums of the squares of the two legs ($a^2 = b^2 + c^2$). What Euclid does that is ingenious, Dunham explains, is to prove this is true using geometry by actually constructing squares on the sides of a right triangle and using his axioms and previous propositions about similar angles and parallel lines shows that these relationships are always true. This Euclid proves in the 47th proposition. In the 48th proposition Euclid does something even more remarkable for his time, which is prove the converse of the theorem, that is, that if a triangle has a side whose square is the sum of the squares of the other two legs, then the triangle is a right triangle.

In the epilogue to the chapter, Dunham returns to one of Euclid's postulates concerning parallel lines. In this postulate, Euclid states that if a line intersecting two other lines forms two angles on the same side which add up to less than two right angles, then the lines will meet if extended and thus are not parallel. This postulate troubles later generations of mathematicians, for it sounds more like a theorem that needs to be proved than a postulate. Other authors attempt to come up with their own consistent definition of parallel lines that can be derived from earlier postulates, but are unsuccessful. It is while working on this problem that the mathematician Carl Friedrich Gauss comes up with a startling discovery.

Gauss sets out to prove that the sum of the angles of a triangle add up to 180 degrees. His strategy to prove this is to assume the opposite, that they do not add up to 180 degrees, and then by showing this to be impossible to prove that they must have 180 degrees. He begins by assuming that the angles add up to more than 180 degrees and indeed finds a contradiction that rules out this possibility. Next he works on the assumption that the angles have fewer than 180 degrees. Gauss is astonished to find that he does not find any logical contradiction in this assumption. Although the geometry that he produces based on this assumption is very different than Euclid's, it is still logically consistent. This "non-Euclidian" geometry is built upon by other mathematicians but is seen largely as nothing more than a pastime, Dunham writes, because it does not seem to have any application to reality, as Euclid's geometry does. It is not until physics catches up in the 20th century, Dunham will explain in a later chapter, that it is recognized that non-Euclidian geometry has a practical use.

Euclid and the Infinitude of Primes

Euclid and the Infinitude of Primes Summary and Analysis

In the third chapter, Dunham looks at the remainder of Euclid's Elements in summary. Euclid goes on to prove other relationships within geometric figures and demonstrates how to construct regular polygons such as a hexagon, a pentagon and a pentadecagon of 15 sides.

Dunham then turns to Euclid's number theory, which examines the nature of whole numbers. While it may seem too basic compared to the more complex theorems of geometry, Dunham warns that number theory is actually very important even in modern math.

Euclid defines even and odd numbers and then moves on to define prime numbers, those special numbers which can only be divided by 1 and themselves. Numbers which can be divided by numbers other than one and themselves he calls composite numbers. Euclid defines the concept of perfect numbers, those numbers with divisors that add up to itself. 6 is a perfect number, for example, because its divisors (besides 6 itself) are 1, 2 and 3 and 1 + 2 + 3 = 6.

Euclid gives considerable attention to the subject of prime numbers and their relationships to other numbers. One of the propositions he proves shows that any composite number is divisible by some prime number, a proof that will be used in the great theorem Dunham has chosen as the main subject of the chapter.

This great theorem is that there exists an infinite number of prime numbers. Euclid states this slightly differently by saying that for any finite group of prime numbers there always exists at least one more. His proposition takes a finite group of prime numbers and adds them together plus one. This creates a new number which is either prime or composite. If the new number is prime, then another prime has been discovered outside the original set of primes and the proof is completed. If the number is composite, then according to the earlier proposition it must have some prime number as a divisor. Euclid shows, however, that this prime divisor cannot be the same as one of the primes in the original group. Therefore, no matter what group of prime numbers one begins with, there is always at least one more prime number that exists which is not part of that group. So the set of all prime numbers is infinite.

In the remainder of Euclid's Elements, he addresses other subjects such as solid bodies and regular polyhedrons. These shapes resonate with Greek philosophers including Plato, who uses them as the building blocks of his theory of the shape of the universe. Euclid's number theory continues to challenge and inspire mathematicians, Dunham explains in the epilogue to the chapter. Other puzzles are suggested by Euclid's theories, such as whether "twin primes," or prime numbers separated by 2 such as 11 and 13, are finite or infinite. This has never been proven one way or the other, Dunham writes. Euclid gives a "recipe" for constructing perfect numbers which always give an even number as a result. He gives no way to find odd perfect numbers, however, and nobody else has been able to either. Whether there are no odd perfect numbers is still not known. Great mathematicians of a later age such as Euler continue to approach these puzzles first suggested in Euclid's great work.

Archimedes' Determination of Circular Area

Archimedes' Determination of Circular Area Summary and Analysis

Archimedes is born in Syracuse on the island of Sicily in the third century BC. He is not only one of the greatest early mathematicians, he is a prolific inventor and scientist. Contemporary accounts of Archimedes portray him as a stereotypical absent-minded genius, Dunham explains, neglecting his everyday responsibilities as he is lost in thought. Archimedes is traditionally described as meeting his death at the hands of a Roman invader after the fall of Syracuse when he refuses to accompany the soldier until he has finished working out a math problem.

Archimedes' great theorem chosen by Dunham regards the measurement of the area of a circle. That the diameter of a circle is always in the same proportion to its circumference is already known in Archimedes' times. While the Greeks did not use the same terminology, this constant relationship is now called π. Furthermore, Euclid proves in his Elements that there is a constant relationship between the area of a circle and the square of its diameter. Archimedes manages to prove that this constant is the same as that between the diameter and the circumference, namely, π.

Archimedes begins by setting out to prove that for any circle, its area is equal to that of a right triangle that has one leg equal to the circle's radius and the other leg equal to the circle's circumference. He does this by a roundabout method of showing that the area of the circle is neither greater than nor less than the area of the triangle and therefore must be equal to it. The area of a triangle is one half the base times the height. Since the base of the triangle in Archimedes' proof is equal to the circumference and the height is the radius, the area of the triangle in his proof is 1/2 the radius times the circumference. And since the circumference can also be expressed as twice the radius multiplied by π, the area is $2\pi r^2/2$, or πr^2, which is the familiar formula for determining the area of a circle.

Archimedes goes one step farther. By constructing multi-sided polygons inside and outside a circle and determining their perimeters, he arrives at a value for π that is between 3 10/11 and 3 1/7, which carry the value to 3.14. This is the first attempt to accurately measure the value of π, and Dunham praises Archimedes for getting so close despite having no way to calculate square roots or other advanced techniques. The measurement of π is a challenge that continues into modern mathematics, Dunham explains in the epilogue to the chapter.

Having successfully tackled the area of a circle, Archimedes goes to the next dimension and looks at the volume and surface area of spheres, cones and cylinders. It is of these examinations that Archimedes himself is most proud. Dunham calls this part of Archimedes' work his true "masterpiece."

Heron's Formula for Triangular Area

Heron's Formula for Triangular Area Summary and Analysis

Archimedes' accomplishments are so pronounced that for a long time nobody approaches the kind of advances he makes in mathematics. Alexandria continues to be a center of thinking and learning, and the chief librarian at the end of the third century BC is a mathematician named Eratosthanes who is best known for having developed a simple way to find prime numbers and for determining the circumference of the Earth. Another Alexandrian mathematician is Apollonius, who develops a work on conics which remains a classic.

Heron is another mathematician at Alexandria. Little is known about the man, but much of his work survives. Heron's work deals largely with practical applications, but he also devises a way to determine the area of a triangle that Dunham chooses as the great theorem of the chapter.

Heron's find the area of a triangle when only the length of the sides are known. Dunham presents Heron's complicated proof over several pages, praising it for its ingenuity and showing how Heron's work draws on that of Euclid.

In the epilogue to the chapter, Dunham shows how Heron's proof also provides an alternate proof of the Pythagorean Theorem. Putting Heron's advances and those of the other mathematicians at Alexandria in historical perspective, Dunham describes the eventual shift in learning from the West to the East, where Arabian scholars are gathering knowledge from around the world. It is in the East that the number system used today is developed and algebra receives its name. Arabian scholars also act as keepers of knowledge allowing the West to rediscover the work of early thinkers during the Renaissance.

Cardano and the Solution of the Cubic

Cardano and the Solution of the Cubic Summary and Analysis

Dunham moves ahead to the 15th century. It is a time of discovery and the advancement of knowledge in Europe. In mathematics, Dunham describes the Italian Luca Pacioli, who writes a treatise that includes a supposition that cubic equations may be impossible to solve. Cubic equations are those in the form $ax^3 + bx^2 + cx + d = 0$.

Pacioli's assessment is taken up as a challenge, Dunham explains. A mathematician named Scipione del Ferro hits upon a method to solve cubic equations by simplifying them into another kind of equation that can be solved. However, Del Ferro does not publish his discovery. Instead he keeps it secret until just prior to his death when he passes it on to a student named Antonio Fior. Fior takes del Ferro's method and challenges another mathematician Niccolo Fontana, also called Tartaglia, to solve 30 problems that Fior is able to solve using del Ferro's method.

Fior is certain only he holds the key to the solutions, but Tartaglia is able to hit upon a solution and meet the challenge. This attracts the attention of the eccentric mathematician Gerolamo Cardano, who approaches Tartaglia asking for the solution to the cubic equation.

Cardano is a strange and superstitious person who leads a storied life and is at one point jailed for heresy. He begs Tartaglia for the solution and Tartaglia eventually relents, but only after Cardano takes an oath never to reveal the secret. Cardano does reveal it to his student Ludovico Ferrari and the two of them make considerable advances that they are unable to publish because they use Tartaglia's method in some part.

Eventually Cardano and Ferrari discover the papers of del Ferro, however, and find the original source of Tartaglia's method. Cardano now feels free to publish his work that uses the method since he has acquired it a second time from a different source. This he does in a work entitled Ars Magna, which includes not only a solution to the cubic equation, but also quartic equations with a term that is raised to the fourth power. Tartaglia is enraged, and a battle or words ensues between him and Ferrari, who answers on behalf of Cardano.

Dunham demonstrates Cardano's solution of the "depressed cubic" as the great theorem of this chapter. In an epilogue, he mentions the solution to the quartic equation again and describes the attempts of mathematicians to solve the next order of equation, the quintic, using algebra. These attempts fail and finally a Norwegian named Niels Abel proves that quintic equations cannot be solved using algebra. The limit of algebraic formulas for solving equations is reached at the quartic.

A Gem from Isaac Newton

A Gem from Isaac Newton Summary and Analysis

In the following century, the center of mathematics shifts from Italy to France and Britain, Dunham explains. Great thinkers such as Francois Viete, Renee Descartes, Blaise Pascal and Pierre de Fermat make great strides in the advancement of mathematics. In Britain, John Napier and Henry Briggs make important discoveries. The largest figure of the period, however, is easily Sir Isaac Newton. Dunham chooses a few of Newton's advances as representatives for the great theorem discussed in this chapter, which is Newton's calculation of π.

Dunham presents a brief biography of Newton from his troubled boyhood through his years at Cambridge as a student and later a professor and into his later years as Warden of the Mint. As a young college student, Newton's genius goes almost unnoticed in an environment that has been overtaken by politics in favor of scholarship. He is encouraged by a professor named Isaac Barrow, however, and his brilliance is soon established. He will eventually make enormous advances in physics and optics as well as mathematics.

Dunham focuses on Newtons use of the binomial theorem and his newly developed method of determining the area under a curve using calculus, calculus having been discovered by Newton as a young man but not published until much later. Using these two tools, Newton estimates π to nine places, closer than any other person has to date. Dunham presents Newton's elegant demonstration of how he arrives at this estimation.

In the epilogue to the chapter Dunham reminds the reader that this remarkable feat of Newton's was completed while he was still a young man. Dunham outlines the later years of Newton's career including his prolific output in all ranges of learning. Dunham hints at the controversy that will take place over the true discovery of the calculus when another mathematician discovers the method independently of Newton. At the time of Newton's death, he is recognized as being alone at the peak of learning, Dunham explains, a position he deserved.

The Bernoullis and the Harmonic Series

The Bernoullis and the Harmonic Series Summary and Analysis

Dunham begins the chapter with a description of the controversy hinted at in the previous chapter concerning the discovery of the calculus. Newton developed the method as a student, but had never published it. Only a few mathematicians are familiar with it from seeing his handwritten notes. A German mathematician, Gottfried Leibniz, visits Britain and sees Newton's notes and is intrigued by this method. He returns to Paris and corresponds with Newton. Leibniz eventually refines and develops a method of calculus of his own and publishes it under his name. British scholars accuse him of plagiarizing Newton. While Leibniz admits the basics are drawn from Newton's work, the method is his own he replies. The controversy rages on on both sides. Today, both Newton and Leibniz are given credit for independently discovering the method.

Two of Leibniz's students are the brothers Jakob and Johann Bernoulli. Jakob, the elder, makes several improvements to the method of calculus of Leibniz, and Johann becomes a great promoter of the method and defender of Leibniz against the claims of his British critics. The great theorem Dunham treats in this chapter is one developed by Jakob and published by Johann with attribution to his brother.

The theorem concerns an infinite series, which at this time in math history is thought of as the sum of a never-ending series of terms. Dunham describes two kinds of these sums. One is the sum of a series such as the numbers 1 + 2 + 3 + 4 and so on. This sum grows ever larger and so is said to "diverge to infinity" (p. 192). Another kind of series has a never-ending string of terms, but sum up to a finite number. The series of "triangular" numbers 1 + 1/2 + 1/6 + 1/10 + 1/15 + 1/21, for example, as shown by Leibniz, approach the sum of 2 (pp. 186-187). These series are said to "converge."

The series that the Bernoullis examine is 1 + 1/2 + 1/3 + 1/4 + 1/5 ... 1/k ... This series, even though it is made up of ever smaller terms like the one that Leibniz sums earlier, does not converge on a finite sum, but, they show, diverges to infinity. It grows at a smaller and smaller rate, but the sum is infinite.

In the epilogue to the chapter, Dunham explains that this proof was not new to mathematics. Two other early mathematicians had arrived at the same results, but their work was unknown to most mathematicians at the time. Dunham describes another series examined by the Bernoullis, one where each successive denominator is a square: 1 + 1/4 + 1/9 + 1/16 ... Johann Bernoulli determines that the sum converges, but is unable to determine what that sum is. It is a more gifted mathematician, Leonhard Euler, a student of the Bernoullis, who will solve this sum and who will be discussed in the following chapter.

The Extraordinary Sums of Leonhard Euler

The Extraordinary Sums of Leonhard Euler Summary and Analysis

Leonhard Euler shows signs of genius at an early age in Switerland, where he is born in 1707. He begins publishing papers as a teen and is appointed by the Russian Czar to the Academy in St. Petersburg at the age of 20. He possesses a remarkable memory and is able to do complicated mathematical operations in his head. Over the course of his life, he produces an incredible amount of work despite his increasing blindness. He dies suddenly in 1783, working on mathematics up until his death.

Euler would certainly have encountered the series $1 + 1/4 + 1/9 + 1/16 \ldots + 1/k^2 \ldots$ from his teacher Johann Bernoulli, Dunham writes, and he sets out to find its sum. He begins by simply adding up the first several terms hoping to find a familiar number. This method does not produce a recognizable result. Using a method that involves both trigonometry and basic algebra, which Dunham demonstrates, Euler reaches the somewhat astonishing conclusion that the sum of the series is $\pi^2/6$.

This is surprising because the series is related to the squares of numbers and the solution is related to π, which is a constant from the measurement of circles. Thus the answer would seem to connect the areas of squares and circles in some way.

In the epilogue to the chapter, Dunham describes Euler's analysis of other series using the same technique. Interestingly, Dunham writes, Euler does not attempt to sum a series where the exponents are odd, such as $1 + 1/2^3 + 1/3^3 + 1/4^3 \ldots 1/k^3 \ldots$ To the present day, Dunham writes, nobody has determined the sum of this series.

A Sampler of Euler's Number Theory

A Sampler of Euler's Number Theory Summary and Analysis

Euler is also remarkable for his contributions to number theory, Dunham writes, and in this chapter he examines some of Euler's theorems concerning prime numbers and the earlier work of Fermat.

Fermat had made several observations about prime numbers which he claimed to have proven, but for which no proof is known at the time of Euler. Euler is able to prove some of Fermat's theorems, including the one called the "little Fermat theorem."

Euler also turns his attention to another of Fermat's claims, which is that $2^{2n} + 1$ is always a prime number. This is true for the first 4 values of n, but Euler discovers that the number produced by the formula where n=5, which is 4,294,967,297, is not prime. He is able to factor it, something no mathematician has been able to before. He thus proves that Fermat's conjecture is wrong, and the method he uses is demonstrated by Dunham.

In the epilogue to the chapter, Dunham introduces Carl Friedrich Gauss who is to take over the mantle of greatness in mathematics from Euler. Like Euler, Gauss shows incredible mathematic ability as a child and enters into a university at a young age. Among his discoveries are a way to construct a regular 17-sided polygon, something thought impossible by the early Greeks and intervening generations. Gauss also made several elemental proofs related to the foundations of algebra.

In Gauss' examinations of the elements of geometry, he constructs a logically consistent system where the angles of a triangle add up to fewer than 180 degrees, as Dunham refers to earlier in the book. Like Newton did before him with his development of calculus, Gauss does not publish these ideas, but keeps them privately in his papers, uncertain how they will be accepted among his colleagues.

The Non-Denumerability of the Continuum

The Non-Denumerability of the Continuum Summary and Analysis

While Euler and his work dominate the 18th century, Dunham writes, no single mathematician stands out in the century that follows. In the second half of the 19th century mathematics continues to exert its independence from the "real" world and centers more and more on the theoretical. Dunham sees a parallel in the world of art and the same time with the new styles of Cezanne, Gaugin and Van Gogh whose paintings are not literal representations of reality.

Dunham writes that while calculus has been in use for nearly two centuries, its foundations have not been completely examined in the late 19th century, especially its assumed notions about "infinitely large" and "infinitely small" quantities. In the intervening years these concepts have been the cause of ridicule from outside mathematics. Mathematicians turn to perfecting the definition of infinite.

Georg Cantor is a Russian-born mathematician living in Germany in the 1860s and 1870s who revolutionizes the concept of infinite sets. He devises a way to compare the relative sizes of infinite sets of items by matching each item in one set to exactly one item in the other. For instance, there is an infinite set of whole numbers 1, 2, 3, 4... and so on. There is also an infinite set of whole numbers that are even: 2, 4, 6, 8... and so on. For each item in the first set, there is exactly one that can be matched with it in the second. Cantor calls sets that can be matched like this "denumerable," and he introduces a symbol to represent the number of items in such a set, $\aleph_0$, "aleph naught."

The great theorem Dunham discisses in this chapter is Cantor's proof that the infinite number of real numbers between 0 and 1 is not denumerable. He calls this interval the "continuum" and introduces another term, c, to describe these sets of numbers that are larger than $\aleph_0$. In further examination, Cantor looks at the sets of rational and irrational numbers and determines that the set of all rational numbers is denumerable, but the set of all irrational numbers is not. In what Dunham calls a "provocative" theorem, Cantor also determines that the set of transcendental numbers such as π is also non-denumerable. Dunham will continue his discussion of Cantor's examination of the infinite in the following chapter.

Cantor and the Transfinite Realm

Cantor and the Transfinite Realm Summary and Analysis

Dunham continues to focus on Cantor and his exploration of the infinite. Cantor develops a way to compare the relative sizes of his cardinal numbers which represent infinite sets, and his efforts are proven by other mathematicians. Cantor has defined two types of transfinite cardinals, $\aleph_0$ and c.

Cantor suspects there are other transfinite cardinals even greater than c, and he sets out to find them. He first tries extending the continuum between 0 and 1 into two dimensions, but finds that the set is still equal to c.

Cantor is finally successful in proving that there are other transfinite cardinals greater than c and his theorem that does so is the great theorem Dunham chooses as the central one of the final chapter. Cantor makes his proof by refining and expanding set theory, which allows for an ever increasing number of sets of sets, and more sets of even those sets and so on forever.

Dunham examines Cantor's religious background and describes Cantor's belief that he was tapping into the nature of God by delving into the infinite. Cantor meets with opposition within the mathematical community against his seemingly incredible theorem. Later in his life h struggles with mental illness. Cantor's theorems shake the foundations of mathematics and force the reexamination of set theory in light of his discoveries.

Dunham ends his work with Cantor and his voyage into the infinite. In a short afterword he concludes with a quote from Bertrand Russell. "Mathematics, rightly viewed, posses not only truth, but supreme beauty." (p. 286)

Characters

Archimedes of Syracuse

Archimedes of Syracuse is a Greek inventor and mathematician of the later third century BC. He lives in Syracuse on the island of Siciiy and is believed to have studied at Alexandria. Archimedes is credited with the invention of the Archimedean screw, a device for raising water from one level to a higher level. He is also attributed with developing several advanced weapons used to defend Syracuse against the invasion of the Romans. Tradition holds that when Syracuse eventually falls to the Romans, Archimedes is killed by a soldier for refusing to come with him until he has completed working out a mathematical problem.

Along with Euclid, Archimedes is one of the greatest mathematicians of ancient times. His great theorem chosen by Dunham as the subject of a chapter is his estimation of the area of a circle and the value of π Using only the rudimentary arithmetic available at the time, Archimedes nonetheless calculates π correctly to 2 decimal places by estimating it to be between 3 1/7 and 3 10/71. This feat is remarkable enough, Dunham suggests, but Archimedes also examines solid figures and reaches the astonishing conclusion that the surface of a sphere is exactly 4 times the area of its greatest circle and that a cylinder constructed around a sphere is 3/2 the surface area and volume of the sphere. This discovery about the relationship between the sphere and the cylinder Dunham says is Archimededes' discovery of which the ancient mathematician is most proud.

After Archimedes, no mathematician is as influential for nearly 2,000 years, Dunham writes. His estimation of π founds a problem that will occupy mathematicians for centuries into the current day, the determination of this crucial ratio to as many places of accuracy as possible.

Euclid

Euclid is a mathematician living in the third and fourth centuries BC, 150 years after Hippocrates. Euclid is the founder of a school of mathematics at Alexandria, the ancient center of learning on the coast of North Africa. He is the author of the Elements, an extensive set of 465 propositions concerning plane and solid geometry and number theory.

Euclid's Elements becomes one of the most studied and widely-dispersed math texts ever, and establishes him as the preeminent mathematician of his day. his influence attracts other great mathematicians to Alexandria, which becomes an important center for the advancement of math theory.

Euclid is revolutionary in his field because of his axiomatic development of his propositions. Beginning with 23 definitions, five postulates and five general axioms, he proceeds to prove 465 increasingly complex propositions, all of which are built on earlier propositions and which can be traced back to the original axioms. While Euclid's arrangement of the postulates and axioms contain some logical steps that do not meet today's more rigorous standards of proof, Dunham writes, all of his propositions have been independently verified and are correct.

Dunham chooses two of Euclid's propositions from the Elements. He devotes one chapter to Euclid's elegant proof of the Pythagorean Theorem from geometry, and from Euclid's number theory he devotes a chapter to Euclid's proof that there are in infinite number of prime numbers.

HIppocrates of Chios

Hippocrates is a Greek mathematician from the fifth century BC. Little is known about him directly, with most of his biographical information coming from later sources that cite his work. He is attributed with authoring the first systematic geometry text which builds up a logical system of proofs based on simple axioms. This text does not exist today, but is thought to have been borrowed from by Euclid in his own text on geometry.

Hippocrates is also well known for his development of a method to find the area of a crescent shape called a "lune" using a compass and straightedge to construct a square of the same area as the lune. This "quadrature of the lune" is the first great theorem treated by Dunham in his book.

Heron

Heron is another of the important mathematicians at Alexandria and is a contemporary of Archimedes. He develops a method for determining the area of a triangle given only the length of the three sides. Dunham examines Heron's theorem for this method in Chapter 5.

Gerolamo Cardano

Gerolamo Cardano is an eccentric and superstitious physician and mathematician living in Italy in the fifteenth and sixteenth centuries. He develops a method to solve cubic equations which are previously thought unsolvable.

Isaac Newton

Isaac Newton is one of the most important figures in the history of science and mathematics. He lives and works in England in the seventeenth century, developing important advances in physics, optics and mathematics. Newton is credited with having developed calculus as a young man, although he does not publish his methods until later in life and only after a German mathematician, Leibniz, publishes his own method of calculus. This sparks a lively and bitter debate over the true founder of this important advance in method. Today, both Newton and Leibniz are considered co-founders of calculus.

Newton's achievements are so numerous, Dunham writes, that it is difficult to take only one as a representative great theorem. Dunham chooses Newton's approximation of π to examine in a chapter devoted to the man. Using his calculus and the binomial theorem that he also develops, Newton is able to calculate π accurately to several decimal places.

Many of Newton's achievements are made when he is a young student at Cambridge. He goes on to become the most respected scientist of his time and is recognized today as one of the most brilliant minds ever.

Jakob and Johann Bernoulli

Jakob and Johann Bernoulli are two brothers who are contemporaries of Leibniz. Though often contentious, they also collaborate in advancing the study of convergent and divergent series.

Leonhard Euler

Euler is one of the greatest mathematical thinkers of all time, Dunham believes, and he devotes two chapters to the eighteenth-century Swiss thinker. Like many of the mathematicians discussed here, Euler is a gifted child with an astounding mathematical ability. As a young man he is admitted to the academy at St. Petersburg in Russia and despite his increasing blindness throughout his life he manages to author several important textbooks. This first great theorem of Euler's that Dunham examines deals with the summation of an infinite series. Euler also makes enormous contributions to the study of number theory and the nature of prime numbers, which Dunham also devotes a chapter to.

Georg Cantor

Georg Cantor is a Russian-born mathematician living in Germany at the end of the nineteenth century. His most prominent work examines the nature of infinite sets of numbers and the nature of infinity itself. Cantor suffers from mental illness possibly aggravated by the opposition he encounters to his controversial theories. He is highly religious, and believes his discoveries about infinity to be revealed to him by God.

Carl Friedrich Gauss

Gauss is a brilliant mathematician of the eighteenth century. At a young age he makes important discoveries in geometry, algebra and number theory and is one of the first to recognize that there are alternative geometries to Euclid's that are also logically consistent. Gauss does not publish his non-Euclidean theories for fear of negative reaction from his colleagues.

PIerre de Fermat

Pierre de Fermat is a French mathematician living in the early years of the seventeenth century. He makes several conjectures about number theory, but even though he claims to have proven them none of his proofs survive. Euler later proves some of Fermat's propositions and disproves others. Fermat is a contemporary of Pascal and Descartes.

Blaise Pascal

Pascal is a seventeenth-century contemporary of Descartes and Fermat. He is a gifted mathematician at a very young age, but gives up the pursuit in favor of theology, for which he is best known today. He later returns to mathematics a few year before his death at age 39.

Gottfried Leibniz

Leibniz is the co-founder if calculus along with Isaac Newton. Leibniz is a "universal genius," mastering many subjects during his life and contributing significantly to all of them. Leibniz combines mathematics and philosophy in an attempt to develop a formal logic.

Pythagoras

Pythagoras is a philosopher and mathematician in ancient Greece who believes that the natural world can be described in units of consistent proportion. The Pythagorean Theorem is named after him.

Thales

Thales is an important figure in early Greek mathematics prior to Hippocrates. He is also an astronomer.

Objects/Places

Pythagorean Theorem

The theorem that expresses the proportions of the sides of a right triangle. The theorem is that the square of the hypotenuse is equal to the sum of the squares of the two perpendicular legs and is written $a^2 = b^2 + c^2$.

Quadrature

The procedure of determining the area of a closed polygon by constructing a square of the same area.

Prime Number

A number that can only be divided evenly by itself and 1.

Binomial Theorem

A theorem refined by Isaac Newton that allows for the expansion of algebraic equations with exponents.

Harmonic Series

The series 1 + 1/2 + 1/3 + 1/4 + ... The sum of the series diverges toward infinity

π

The Greek letter "pi." π is the ratio between the diameter and the circumference of a circle.

Transfinite Cardinal

A number used to describe types of infinite sets.

Cubic Equation

A function of the form $ax^3 + bx^2 + cx + d$ where a is not equal to zero.

Continuum

Georg Cantor's term for all the numbers between to integers, such as between 0 and 1.

Elements

A book by Euclid that develops complex proofs of geometric relationships built upon basic axioms.

Themes

The Measurement of π

One of the important mathematical themes that runs throughout Dunham's book is the measurement of the ratio π. He documents a passage in the Bible that indicates that it has long been known that a circle's diameter is roughly one-third its circumference. He charts the ever-increasing accuracy of the estimation of the number, using it as a kind of measuring stick to gauge how advanced the state of mathematics is at a given point in history. It is also used as a measurement of the ability of an individual scholar.

Archimedes is the first mathematician Dunham describes who reaches an estimate of π to two decimal places of accuracy, although Archimedes did not use a decimal system. Dunham praises Archimedes for having come so close using only the computational tools that were available to him. For centuries afterward, Archimedes' method of measuring the perimeter of multi-sided polygons that approach circularity is the method used to advance the measurement of π. As the ability to compute the square roots necessary to use the method advances and decimal system of notation is introduced, the number of places of accuracy increases, but the method is essentially the same one Archimedes uses.

A breakthrough occurs when Isaac Newton discovers a new way to calculate the area under a curve using a method which is eventually to be known as integral calculus. Newton's method is able to reach greater accuracy with fewer steps. Newton's methods turn out to be crucial to many areas of mathematics, but Dunham chooses his calculation of π as representative as it provides a clear benchmark of progress.

Later, π appears again in the number theory of Leonhard Euler, who is astonished to find the number appearing in the sums of infinite series. Finally, Georg Cantor determines that as far as numbers go, π is perhaps not as unique as once thought. He shows that transcendental numbers like π are among the most abundant of all types of numbers. Cantor has delved deeper into the nature of π than anyone before. The suggestion is that one way to measure the advancement of mathematics is to assess an era's relationship with this strange and elusive number.

Practical and Theoretical Mathematics

Mathematics probably has its roots in the practical, Dunham writes. Earliest civilized humans would have had need to count things such as livestock and measure things like fields. Many of the earliest mathematicians are also inventors, such as Archimedes who develops machines and weapons as well as advancing the field of geometry, and Heron who specializes in building and engineering.

Even the more theoretical aspects of early geometry are assumed to have practical applications among the early Greeks, Dunham explains. Plato develops a cosmology of the universe that is made up of the regular solids even though these shapes cannot actually be seen in nature.

It is with Euclid, and also in later generations' reactions to Euclid, that Dunham finds the seeds of theoretical mathematics. Euclid's geometry does indeed describe the actual world. His plane figures can be drawn on any flat surface and reproduced using compass and straightedge in the real world. Euclid's method, however, while not entirely new, is the most complete and influential example of an axiomatic system that is self-contained. It does not rely on the natural world for verification, but on a set of definitions and axioms that are an intrinsic part of the system.

One of the propositions that Euclid relies upon in his work is that the angles of all triangles add up to 180 degrees. This is how triangles behave in the "real" world, but later mathematicians like Beltrami and Gauss begin to imagine a geometry where triangles do not contain 180 degrees and find they are still able to construct logically consistent axiomatic systems. In the case of Gauss, he keeps his discovery to himself, afraid that his conclusions will be judged too radical by his colleagues. This fear of theoretical mathematics is part of this theme that Dunham traces. He documents other mathematicians like Newton and Cantor who hide their wilder conclusions from the mathematical community.

In hindsight, Dunham suggests, these fears are ironic, for many of the ideas which begin as theoretical exercises turn out to have practical uses when science and technology "catch up" to the theory. Advanced physics currently makes practical use of the type of non-Euclidean geometry that Gauss hid from view.

Dunham also chooses to cap his book with a lengthy discussion of the work of Georg Cantor who is the first mathematician to "count" the infinite by defining different orders of transfinite numbers. Like other theorists before him, Cantor encounters oppositions to his views and in his case feels persecuted by his colleagues. Nevertheless he persists as he believes God is revealing the nature of infinity to him. His is an almost purely metaphysical, religious approach to mathematics. Thus Dunham has charted the path of mathematics from the purely practical function of counting sheep and measuring fields to the supernatural inquiry into the nature of infinity, and, perhaps, in Cantor's view, the nature of God.

The Process and Progress of Mathematics

Dunham describes mathematics as an advancing field of inquiry, that is, it is almost continually being refined and improved upon. By placing his choice of twelve great theorems in a historical context, Dunham shows the processes by which the mathematical community moves the inquiry forward.

Dunham portrays the early mathematical community centered on Alexandria as a collegial one where scholars correspond with one another and share their knowledge. Some of them, like Euclid, are teachers whose profession calls on them to share their discoveries. This seems to ensure the survival of some of these early theorems, but on the other hand there is no great leap forward in mathematics for many centuries following the early Greeks.

The mathematical community described by Dunham since the sixteenth century seems very different. Mathematicians are often secretive with their discoveries, either out of a sense of competition with their colleagues or out of fear that their more esoteric theories may damage their reputations. Scholars regularly issue open challenges to one another and slight their colleagues in their published works. Dunham describes the protracted bitter rivalry between the supporters of Newton and Leibniz over the true founder of calculus when either man's achievements apart from this one would ensure them each eternal places among the world's most gifted minds. This rivalry and competition seems to spur many minds to action, however, and many advances are made from it. On the other hand, this atmosphere of rivalry beats down great thinkers like Georg Cantor, who suffers from the idea that he is being persecuted for his ideas.

Dunham documents these processes but does not judge them. Implied, however, is that rivalry and competition stimulate the advancement of mathematics, but that the sword is double-edged.

Style

Perspective

Dunham is a college professor, and he writes from the perspective of an historian and educator. His goal is not to instruct in mathematics, but to place math in a historical context and treat it as other themes that are traditionally followed in historical research such as art, music and science.

Dunham clearly believes mathematics deserves equal examination in a historical context, and he structures his book to emphasize how each theorem he demonstrates is a unique product of its author and its time. He is almost reverential toward some of his subjects such as Leonhard Euler, Georg Cantor and Archimedes, all of whom are given large portions of the book.

Dunham is also writing from the perspective of a modern mathematician who has the benefit of the intervening years of study since the theorems he describes first appeared. He is able to identify where earlier mathematicians made assumptions that would not be made today and identify them to his readers, explaining whether or not these assumptions change the validity of the theorem and how they are improved upon by later generations of scholars.

As an educator, Dunham is adept at offering the appropriate amount of information for his purpose. He indicates when a particular mathematical theorem or operation is beyond the scope of his book and demonstrates what are sometimes very complex ideas in a way that is sufficient to show his readers the importance of a certain theorem even if the reader is not completely familiar with the more advanced features of it.

Tone

Dunham adopts a light tone through much of the work, one suitable for the type of book he intends to write and the audience he hopes to reach. The tone is similar to that of a casual lecture designed to give an overview of a subject while maintaining the interest of the audience.

The biographies Dunham supplies of various figures from mathematical history include mainly the most interesting highlights of their lives including any quirks or unusual occurrences. He draws basic comparisons between many of them based on these broad portraits such as the socially isolated and obsessed genius figures of Newton and Cantor, or the stereotypical absent-minded genius of Archimedes as a kind of shorthand for his reader to flesh out his basic descriptions.

When demonstrating mathematic theorems, Dunham's tone is straightforward but not pedantic. He provides alternate examples of complex aspects of many of the theorems and does not assume advanced knowledge of mathematics in his reader. He tempers his sometimes lengthy explanations with sympathetic aides to the reader in which he acknowledges that the argument or example turns on a particularly subtle point.

Dunham also uses humorous asides to draw his reader into his narrative, sometimes poking fun at the beliefs and practices of his subjects, such as the naive imaginings of the Pythagoreans or the paranoid superstitions of Cardano.

Structure

Dunham divides his book into 12 chapters, each treating a specific mathematical theorem. The chapters are arranged in chronological order, covering from early Greek times into the 19th century. Dunham includes a brief preface and afterword.

Dunham divides each chapter into subsections, beginning with discussion of the historical context in which the theorem treated by the chapter appears. He also provides biographical information on the theorem's author and background on other mathematicians whose work is related to it. In cases where the theorem being presented requires an understanding of other theorems, Dunham includes separate subsections treating these topics.

Each chapter includes a subsection entitled "The Great Theorem" in which Dunham presents the theorem and shows its proof. He ends each chapter with an epilogue, which further places the theorem and its author in perspective, particularly as they relate to the theorems that follow in later chapters, and in the context of modern mathematics.

Dunhams overall organization is chronological, and he attempts not to include every great mathematical theorem, but to connect a dozen of them through time to represent how ideas about mathematics have changed and to discuss how the frontier of advanced mathematics is influenced by history.

Quotes

"It is obvious that the ancient Greeks were enthralled by the symmetries, the visual beauty, and the subtle logical structure of geometry. Particularly intriguing was the manner in which the simple and elementary could serve as foundation for the complex and intricate." p. 11

"What Euclid did that established him as one of the greatest names in mathematics history was to write the Elements. This work had a profound impact on Western thought as it was studied, analyzed, and edited for century upon century, down to modern times. It has been said that of all books from Western civilization, only the Bible has received more intense scrutiny than Euclid's Elements." p.30

"One of the genuine attractions of number theory is that conjectures simple enough to be understood by elementary school students nonetheless have been immune to the efforts of generations of the world's best mathematicians. It seems an especially perverse feature of this corner of mathematics." p. 81

"One legacy of Archimedes' Measurement of a Circle was the quest for ever more precise estimates of the critical constant we call pi. The importance of this ratio had been recognized long before Archimedes, although it was he who first subjected it to a scientific scrutiny." p. 106

"Archimedes cast a very long shadow across the mathematical landscape. Subsequent mathematicians of the classical period left their marks, but none even remotely measured up to the great Syracusan, and observation that became ever more obvious with the fall of Greek civilization and the simultaneous rise of Rome." p. 113

"Without question, the last decades of the fifteenth century marked a time of great intellectual excitement in Europe. Western civilization had clearly awakened from the slumber of the Middle Ages." p. 133

"One need only recall the world view of Cardano from a century before - a peculiar blend of hard science and the most outlandish superstition. At that time, the world was seen largely as an irrational place, with supernatural agencies intervening in everything from the appearance of comets to the daily calamities of life. Newton, with his clockwork universe, removed the supernatural from Nature." p. 182

"While the solitary Isaac Newton was changing the face of mathematics from his Cambridge rooms, mathematicians on the European Continent were far from idle. Influenced by the work of Descartes, Pascal, and Fermat, continental mathematics flourished during the latter half of the seventeenth century. Its greatest practitioner by far was Gottfried Wilhelm Leibniz." p. 184

"Euler could be convicted of giving insufficient attention to the logical foundations of his arguments. Yet such criticisms barely tarnish his reputation. Even though his approach to infinite series was naive, all of these wonderful sums have been subsequently verified by today's higher standards of logical rigor." p. 222

"The fundamental theorem of algebra - the result that establishes the complex numbers as the ultimate realm for factoring polynomials - thus remained in a very precarious state. D'Alembert had not proved it; Euler had given only a partial proof. It was obviously in need of major attention to resolve its validity once and for all." p. 238

"If the 1800s did not belong to a single mathematician, they did have a few overriding themes. It was the century of abstraction and generalization of a deeper analysis of the logical foundations of mathematics that underlay the wonderful theories of Newton and Leibniz and Euler." p. 245

"It is an understatement to say that this conclusion, along with all of Cantor's profound results about the infinite, generated an outcry of opposition. Surely, he had pushed mathematics into unexplored territory where it began to merge into the realms of philosophy and metaphysics." p. 278

Topics for Discussion

What role has religion played in mathematics according to Dunham?

How do social influences affect the development of mathematic theorems?

Math and science have always been closely related. How does Dunham characterize this relationship?

How does Dunham compare the pursuit of mathematics with art? Is his comparison apt?

How does Dunham describe the shifting line between practical and theoretical mathematics?

Does placing a mathematical theorem in historical perspective help explain the theorem, or does math stand free from history?

Does competition and rivalry drive or hinder advances in mathematics according to Dunham's account?